Let a, b, c be non-negative numbers. Prove that

$$\frac{a^5}{a^4+b^4}+\frac{b^5}{b^4+c^4}+\frac{c^5}{c^4+a^4} \ge \frac{a+b+c}{2} .$$

Proof. Suppose, without loss of generality, that $c=\max\{a,b,c\}$. The inequality is equivalent to

$$(\frac{a^5}{a^4+b^4}-\frac{a}{2})+(\frac{b^5}{b^4+c^4}-\frac{b}{2})+(\frac{c^5}{c^4+a^4}-\frac{c}{2}) \ge 0 ,$$

or

$$\frac{a(a^4-b^4)}{a^4+b^4}+\frac{b(b^4-c^4)}{b^4+c^4}+\frac{c(c^4-a^4)}{c^4+a^4} \ge 0 .$$

Since

$$\frac{a(a^4-b^4)}{a^4+b^4}-\frac{b(a^4-b^4)}{a^4+b^4}=\frac{(a-b)(a^4-b^4)}{a^4+b^4}\ge 0,$$

it suffices to show that

$$\frac{b(a^4-b^4)}{a^4+b^4}+\frac{b(b^4-c^4)}{b^4+c^4}+\frac{c(c^4-a^4)}{c^4+a^4} \ge 0 .$$

Since

$$\frac{b(a^4-b^4)}{a^4+b^4}+\frac{b(b^4-c^4)}{b^4+c^4}=\frac{2b^4(a^4-c^4)}{(a^4+b^4)(b^4+c^4)} .$$

we derive that the last inequality is equivalent to

$$(c^4-a^4)(c-b)[a^4(2b^4+b^3c+b^2c^2+bc^3+c^4)+b^4c(c^3-b^3-b^2c-bc^2)] \ge 0 . \tag{1}$$

Since $(c^4-a^4)(c-b)\ge 0$, it is enough to show that

$$a^4(2b^4+b^3c+b^2c^2+bc^3+c^4)+b^4c(c^3-b^3-b^2c-bc^2)\ge 0 . \tag{2}$$

Let $x=\frac{b}{c}$, $0<x\le 1$. We notice that (2) is true for $b^3+b^2c+bc^2\le c^3$, that is

$$x^3+x^2+x\le 1.$$

In addition, (2) is true for $ac\ge b^2$. To prove this fact, it suffices to prove (2) for $a=\frac{b^2}{c}$, that is

$$2x^8+x^7+x^6+x^5+x^4-x^3-x^2-x+1\ge 0 .$$

Case $0<x\le\frac{\sqrt{5}-1}{2}$. We have

$$2x^8+x^7+x^6+x^5+x^4-x^3-x^2-x+1>x^5+x^4-x^3-x^2-x+1=(1-x-x^2)(1-x^3)\ge 0$$

Case $\frac{\sqrt{5}-1}{2}<x\le 1$. Since $x^8\ge x^9$, it suffices to show that

$$x^9+x^8+x^7+x^6+x^5+x^4-x^3-x^2-x+1\ge 0 .$$

This inequality is equivalent to

$$\frac{x^2}{x^2+x+1}\ge x^3-x^6-x^9 .$$

Since

$$27(x^3-x^6-x^9)=5-(3x^3-1)^2(3x^3+5)\le 5,$$

it suffices to show that

$$\frac{x^2}{x^2+x+1}\ge\frac{5}{27}.$$

This inequality reduces to $x\ge\frac{5+\sqrt{465}}{44}$, and it is true because $x>\frac{\sqrt{5}-1}{2}>\frac{27}{44}>\frac{5+\sqrt{465}}{44}$.

Based on obtained results, from here on we will assume that $x^3+x^2+x>1$ and $b^2>ac$. The last condition implies $b>a$, therefore $0\le a<b\le c$. By permutation, from (1) we find that the given cyclic inequality is also valid under the condition

$$(a^4-b^4)(a-c)[b^4(2c^4+c^3a+c^2a^2+ca^3+a^4)+c^4a(a^3-c^3-c^2a-ca^2)]\ge 0. \qquad (3)$$

Since $(a^4-b^4)(a-c)>0$, it is enough to show that

$$b^4(2c^4+c^3a+c^2a^2+ca^3+a^4)+c^4a(a^3-c^3-c^2a-ca^2)\ge 0.$$

Since $b^2>ac$, it suffices to prove this inequality for $b^2=ac$, that is

$$x^5+x^4+2x^3+x-1\ge 0.$$

Indeed, we have

$$x^5+x^4+2x^3+x-1=(x^2+1)(x^3+x^2+x-1)>0,$$

and this completes the proof. Equality occurs only if $a=b=c$.