TMO 16th Problems and Solutions

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On May 18-22, 2019, the 16th Thailand Mathematical Olympiad (TMO) was held in Silpakorn University. Here I complied all the problems and the solutions from the TMO. The exam translations provided in this document are not official.

Corrections and comments are welcome!

Contents

Sunday, May 19th, 2019

Problem 1. Let $ABCDE$ be a convex pentagon such that $\angle AEB = \angle BDC = 90^\circ$ and line AC bisect both angles $\angle BAE$ and $\angle DCB$. Let the circumcircle of $\triangle ABE$ meet the line AC again at point P.

(i) Prove that P is the circumcenter of $\triangle BDE$.

(ii) Prove that points A, C, D, E lie on a circle.

Problem 2. Let a, b be two different positive integers. Suppose that a, b are relatively prime. Prove that $\dfrac{2a(a^2+b^2)}{a^2-b^2}$ is not an integer.

Problem 3. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(x + yf(x) + y^2) = f(x) + 2y$ for every $x, y \in \mathbb{R}^+$.

Problem 4. A rabbit initially stands at the position 0, and repeatedly jumps on the real line. In each jump, the rabbit can jump to any position corresponds to an integer but it cannot stand still. Let $N(a)$ be the number of ways to jump with a total distance of 2019 and stop at the position a. Determine all integers a such that $N(a)$ is odd.

Problem 5. Let a, b, c be positive reals such that $abc = 1$. Prove the inequality

$$\frac{4a-1}{(2b+1)^2} + \frac{4b-1}{(2c+1)^2} + \frac{4c-1}{(2a+1)^2} \geqslant 1.$$

Time: 4 hours and 30 minutes

Each problem is worth 7 points.

Language : English

Day : 2

Monday, May 20th, 2019

Problem 6. Determine all function $f : \mathbb{R} \to \mathbb{R}$ such that $xf(y)+yf(x) \leqslant xy$ for all $x, y \in \mathbb{R}$.

Problem 7. Let $A = \{-2562, -2561, ..., 2561, 2562\}$. Prove that for any bijection (1-1, onto function) $f : A \to A$,

$$\sum_{k=1}^{2562} |f(k) - f(-k)| \text{ is maximized if and only if } f(k)f(-k) < 0 \text{ for any } k = 1, 2, ..., 2562.$$

Problem 8. Let ABC be a triangle such that $AB \neq AC$ and ω be its circumcircle. Let I be the center of the incircle of $\triangle ABC$ which touches BC at D. The circle with diameter AI intersects ω again at K. Line AI meets ω again at M. Prove that points K, D, M are colinear.

Problem 9. A *chaisri* figure is a triangle which the three vertices are vertices of a regular 2019-gon. Two different chaisri figure may be formed by different regular 2019-gon.

A *thubkaew* figure is a convex polygon which can be dissected into multiple chaisri figure where each vertex of a dissected chaisri figure does not necessarily lie on the border of the convex polygon.

Determine the maximum number of vertices that a thubkaew figure may have.

Problem 10. Prove that there are infinitely many positive odd integer n such that $n! + 1$ is composite number.

Time: 4 hours and 30 minutes

Each problem is worth 7 points.

§1 Day 1 Solutions

§1.1 Solution to Problem 1

Problem 1. Let $ABCDE$ be a convex pentagon such that $\angle AEB = \angle BDC = 90^\circ$ and line AC bisect both angles $\angle BAE$ and $\angle DCB$. Let the circumcircle of $\triangle ABE$ meet the line AC again at point P.

(i) Prove that P is the circumcenter of $\triangle BDE$.

(ii) Prove that points A, C, D, E lie on a circle.

Clearly P is the projection from B to AC thus quadrilaterals $ABPE$ and $CBPD$ are cyclic. This implies that $PB = PD = PE$, completing (i).

For (ii), we just observe that

$$\angle CAE + \angle CDE = \angle PBE + (90^\circ + \angle BDE) = 180^\circ.$$

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§1.2 Solution to Problem 2

Problem 2. Let a, b be two different positive integers. Suppose that a, b are relatively prime. Prove that $\frac{2a(a^2+b^2)}{a^2-b^2}$ is not an integer.

The main claim is

Claim (Gcd bash)

We have $\gcd(a^2-b^2, a(a^2+b^2)) \mid 2$.

Proof. Let $d \mid a^2-b^2$ and $d \mid a(a^2+b^2)$. We get

$$\begin{aligned} d \mid a(a^2+b^2)+a(a^2-b^2) &\implies d \mid 2ab^2 \\ d \mid 2a(a^2+b^2) &\implies d \mid 2a^3 \\ d \mid a^2-b^2 \mid 2(a^4-b^4) &\implies d \mid 2b^4 \end{aligned}$$

thus $d \mid 2\gcd(a,b)^4 = 2$ so we are done. □

By the claim, we get $a^2-b^2 \mid 4$ which is a clear contradiction.

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§1.3 Solution to Problem 3

Problem 3. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(x + yf(x) + y^2) = f(x) + 2y$ for every $x, y \in \mathbb{R}^+$.

The answer is $f(x) = 2\sqrt{x + c}$ for any constant $c \geqslant 0$. It's straightforward to verify that they are all work.

Now we present several solutions which show that these are all possible solutions.

Solution 1 (Substitution, Pitchayut): Plug in $(x, y) = \left(a, \frac{\sqrt{f(a)^2+4b}-f(a)}{2}\right)$ gives

$$f(a+b) = \sqrt{f(a)^2 + 4b} \implies f(a+b)^2 = f(a)^2 + 4b$$

This is enough to conclude that $f(x)^2 = 4x + c$ for some constant c. This implies the set of solutions mentioned above.

Solution 2 (Squaring, Nithid): Squaring the entire equation gives

$$f(x + yf(x) + y^2)^2 = f(x)^2 + 4(yf(x) + y^2).$$

By Intermediate Value Theorem, function $g(y) := yf(x)+y^2$ is surjective on $\mathbb{R}^+$ as $\lim_{y\to 0} g(y) = 0$ and $\lim_{y\to\infty} g(y) = \infty$. Thus we get

$$f(x+t)^2 = f(x)^2 + 4t$$

for any $x, t \in \mathbb{R}^+$. This is enough to conclude the solution.

Solution 3 (Injectivity): First, we prove that f is injective. Suppose that $f(x + t) = f(x)$ for some $x, t \in \mathbb{R}^+$. Arguing as in above solutions, we can choose appropriate y such that $yf(x) + y^2 = t$. Using this choice of x, y in the equation gives $y = 0 \implies t = 0$ which is contradiction.

Now it's easy to finish. Plug in $y = \frac{f(t)}{2}$ gives

$$f\left(x + \frac{f(x)f(t)}{2} + \frac{f(t)^2}{4}\right) = f(x) + f(t) = f\left(t + \frac{f(x)f(t)}{2} + \frac{f(x)^2}{4}\right).$$

By injectivity, $4x + f(t)^2 = 4t + f(x)^2$ so we are done.

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§1.4 Solution to Problem 4

Problem 4. A rabbit initially stands at the position 0, and repeatedly jumps on the real line. In each jump, the rabbit can jump to any position corresponds to an integer but it cannot stand still. Let $N(a)$ be the number of ways to jump with a total distance of 2019 and stop at the position a. Determine all integers a such that $N(a)$ is odd.

Solution 1 (Generating Function, Pitchayut): Consider the quantity

$$T = (x + x^2 + x^3 + ...) + (y + y^2 + y^3 + ...) = \frac{x}{1-x} + \frac{y}{1-y}$$

and define generating functions

$$F(x, y) = 1 + T + T^2 + ...$$

It's clear that the coefficient of $x^a y^b$ in F equals to the number of ways to jump with a total distance of $a + b$ and arrive at position $a - b$. (i.e. variable x corresponds to positive jumps and variable y corresponds to negative jumps).

Now we evaluate $F(x, y)$ in $\pmod 2$. To do this, let $G(x, y) = 1 - T + T^2 - T^3 + ...$ so that $G \equiv F \pmod 2$ and

$$G(x, y) = \frac{1}{1+T} = \frac{1}{1 + \frac{x}{1-x} + \frac{y}{1-y}} = \frac{(1-x)(1-y)}{1-xy}$$

Thus, we have

$$G(x, y) = (1 - x - y + xy)(1 + (xy) + (xy)^2 + (xy)^3 + ...)$$

It's clear that all odd coefficients are in form $x^n y^{n+1}$ and $x^{n+1} y^n$, which corresponds to $N(1)$ and $N(-1)$. Thus the answer is $\boxed{\{1, -1\}}$.

Solution 2 (Combinatorial, Official): Encode each positive jump by the corresponding number of + and encode each negative jump by the corresponding number of -. We also seperate each jump by |. For instance,

$$\texttt{++|+++|---} \implies 0 \xrightarrow{+2} 2 \xrightarrow{+3} 5 \xrightarrow{-3} 2.$$

Clearly, in $N(a)$, we must have $\frac{2019+a}{2}$ +'s and $\frac{2019-a}{2}$ -'s. We also note that a | must be inserted between +-.

Now, fix a sequence consisting many + and -. Call a sequence *bad* if and only if there are odd number of ways to insert |.

Claim

The only bad sequences are `+-+-+...+` and `-+-+-...-`

Proof. If m, n denote the number of consecutive `++` and `--` respectively. Then clearly the number of ways to insert `|` is precisely 2^{m+n}. Thus the sequence is bad if and only if there are no `++` and `--` at all so we are done. $\square$

The two bad sequences correspond to $N(1)$ and $N(-1)$ thus the answer is $\{1, -1\}$.

§1.5 Solution to Problem 5

Problem 5. Let a, b, c be positive reals such that $abc = 1$. Prove the inequality

$$\frac{4a-1}{(2b+1)^2} + \frac{4b-1}{(2c+1)^2} + \frac{4c-1}{(2a+1)^2} \geqslant 1.$$

Add one to each term and divide by 4. This is equivalent to

$$\sum_{\text{cyc}} \frac{b^2+b+a}{(2b+1)^2} \geqslant 1$$

Now we can use Cauchy Schwarz in form $(b^2+b+a)(1+b+\frac{1}{a}) \geqslant (b+b+1)^2$. Thus it suffices to prove that

$$\sum_{\text{cyc}} \frac{1}{b+\frac{1}{a}+1} \geqslant 1.$$

In fact, it turns out to be an equality. The cleanest way to verify that is to substitute $a = \frac{x}{y}, b = \frac{z}{x}, c = \frac{y}{z}$ and see that each term is equal to $\frac{x}{x+y+z}$.

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§2 Day 2 Solutions

§2.1 Solution to Problem 6

Problem 6. Determine all function $f : \mathbb{R} \to \mathbb{R}$ such that $xf(y) + yf(x) \leqslant xy$ for all $x, y \in \mathbb{R}$.

The answer is $f(x) = \frac{x}{2}$ which clearly works.

Define $g(x) = f(x) - \frac{x}{2}$. Then the given equation is equivalent to $xg(y) + yg(x) \leqslant 0$. We aim to show that $g \equiv 0$.

Plugging in $x = y$ gives $xg(x) \leqslant 0$. Thus plugging in $y = -x$ gives

$$xg(-x) + \underbrace{(-xg(x))}_{\geqslant 0} \leqslant 0 \implies xg(-x) \leqslant 0$$

Replacing x by $-x$ gives $xg(x) \geqslant 0$ for any $x \in \mathbb{R}$. This is enough to conclude that $g(x) = 0$ for any $x \neq 0$.

Seeking $g(0)$, we just drop $x = 0$ and we are done.

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§2.2 Solution to Problem 7

Problem 7. Let $A = \{-2562, -2561, ..., 2561, 2562\}$. Prove that for any bijection (1-1, onto function) $f : A \to A$,

$$\sum_{k=1}^{2562} |f(k) - f(-k)| \text{ is maximized if and only if } f(k)f(-k) < 0 \text{ for any } k = 1, 2, ..., 2562.$$

Clearly we can swap $f(k)$ and $f(-k)$ without any trouble. Thus WLOG $f(k) > f(-k)$ for any $k = 1, 2, ..., 2562$. The expression evaluates to

$$f(1) + f(2) + ... + f(2562) - f(-1) - f(-2) - ... - f(-2562).$$

Evidently it's minimized when $\{f(1), f(2), ..., f(2562\} = \{1, 2, ..., 2562\}$ and $\{f(-1), f(-2), ..., f(-2562)\} = \{-1, -2, ..., -2562\}$ which is basically the problem's condition.

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§2.3 Solution to Problem 8

Problem 8. Let ABC be a triangle such that $AB \neq AC$ and ω be its circumcircle. Let I be the center of the incircle of $\triangle ABC$ which touches BC at D. The circle with diameter AI intersects ω again at K. Line AI meets ω again at M. Prove that points K, D, M are colinear.

Solution 1 (Spiral Similarity): Let the incircle touches AC, AB at E, F. Then just notice the spiral similarity $\triangle KBF \overset{+}{\sim} \triangle KCE$ thus

$$\frac{KB}{KC} = \frac{BF}{CE} = \frac{BD}{DC}$$

or KD bisects $\angle BKC$. This immediately implies K, D, M are colinear.

Solution 2 (Inversion): Again, let E, F be the other two intouch points. Perform inversion around the incircle. We deduce the following facts.

- $\triangle A'B'C'$ is medial triangle of $\triangle D'E'F'$.
- I is orthocenter of $\triangle A'B'C'$.
- M' is reflection of I across $B'C'$.
- K' is foot from D' to $E'F'$.

This means points $\{K', D'\}$ and $\{I, M'\}$ are symmetric across $B'C'$. So $K'D'M'I$ is isosceles trapezoid which obviously cyclic. Inverting back, we find that K, D, M are colinear.

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§2.4 Solution to Problem 9

Problem 9. A *chaisri* figure is a triangle which the three vertices are vertices of a regular 2019-gon. Two different chaisri figure may be formed by different regular 2019-gon.

A *thubkaew* figure is a convex polygon which can be dissected into multiple chaisri figure where each vertex of a dissected chaisri figure does not necessarily lie on the border of the convex polygon.

Determine the maximum number of vertices that a thubkaew figure may have.

The answer is $\boxed{4038}$.

To see the bound, note that each angle must be multiple of $\frac{\pi}{2019}$. Thus each angle has magnitude at most $\frac{2018\pi}{2019}$. Thus if the n-gon works, then

$$\pi(n-2) \leqslant \tfrac{2018\pi}{2019} \cdot n \implies n \leqslant 4038.$$

For the construction, take a regular 4038-gon and draw a line connecting the center to each of the 4038 vertices.

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§2.5 Solution to Problem 10

Problem 10. Prove that there are infinitely many positive odd integer n such that $n! + 1$ is composite number.

For each odd n, either n or $n! - n$ works. To see why, let $n! + 1 = p$ be a prime. Then by a variant of Wilson's Theorem,

$$n!(p-1-n)! \equiv (-1)^{n-1} \pmod p \implies (n!-n)! = (p-1-n)! \equiv -1 \pmod p$$

thus $p \mid (n!-n)! + 1$ so $n! - n$ works.
