SOME COSINE RELATIONS AND THE REGULAR HEPTAGON

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1. INTRODUCTION

The ancient Greek mathematicians sought to construct, by use of straight edge and compass only, all regular polygons. They had no difficulty with regular polygons having 3, 4, 5 and 6 sides, but the 7-sided heptagon eluded all their attempts. Today we know that this construction is impossible. A remarkable elementary proof of this impossibility can be found in [1], requiring only precalculus mathematics. In Figure 1, we see the regular heptagon with its central angle $\alpha = 2\pi/7 \approx 51.42857143°$. Cosines of the central angles of the first few regular polygons are expressed in closed form as

$$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2},\ \cos\left(\frac{2\pi}{4}\right) = 0,\ \cos\left(\frac{2\pi}{3}\right) = \frac{\sqrt{5}-1}{4},\ \text{and}\ \cos\left(\frac{2\pi}{3}\right) = \frac{1}{2}.$$

But, $\cos\left(\frac{2\pi}{7}\right)$ cannot be expressed in closed form this simply. We need complex numbers and cube roots as shown in (1). In this paper we will be concerned with $\cos\alpha$, $\cos 2\alpha$ and $\cos 3\alpha$, where $\alpha = 2\pi/7$, which have the approximate values 0.6234898019, -0.222520933956, and -0.9009688679 respectively. We will show that exact values are given by the cumbersome closed form expressions

$$\cos(\alpha) = -\frac{1}{6} + \frac{1}{6}\left(\frac{7}{2}\right)^{1/3} \sqrt[3]{1+3i\sqrt{3}} + \frac{1}{3}\left(\frac{7}{2}\right)^{2/3} \frac{1}{\sqrt[3]{1+3i\sqrt{3}}}, \tag{1}$$

$$\cos(2\alpha) = -\frac{1}{6} - \frac{1}{12}\left(\frac{7}{2}\right)^{1/3} \left(1+i\sqrt{3}\right)\sqrt[3]{1+3i\sqrt{3}} - \frac{1}{6}\left(\frac{7}{2}\right)^{2/3} \frac{1-i\sqrt{3}}{\sqrt[3]{1+3i\sqrt{3}}}, \tag{2}$$

and

$$\cos(3\alpha) = -\frac{1}{6} - \frac{1}{12}\left(\frac{7}{2}\right)^{1/3} \left(1-i\sqrt{3}\right)\sqrt[3]{1+3i\sqrt{3}} - \frac{1}{6}\left(\frac{7}{2}\right)^{2/3} \frac{1+i\sqrt{3}}{\sqrt[3]{1+3i\sqrt{3}}}. \tag{3}$$

(In general, there are three cube roots of any number. By $\sqrt[3]{1+3i\sqrt{3}}$ we mean the root in the first quadrant which is approximately

$1.561271168 + i\,0.773964671$.) Notice that cube roots of complex numbers are used to express these functions.

While $\cos\alpha$, $\cos 2\alpha$ and $\cos 3\alpha$, where $\alpha = 2\pi/7$, are not constructible, it is surprising that certain simple algebraic combinations of them are rational numbers. These include

$$\cos\alpha + \cos 2\alpha + \cos 3\alpha = -1/2\,, \tag{4}$$

$$\sec\alpha + \sec 2\alpha + \sec 3\alpha = -4\,, \tag{5}$$

$$\text{and } \cos\alpha\cos 2\alpha\cos 3\alpha = 1/8\,. \tag{6}$$

We will see that all three of the above are easy to derive.

The next two expressions were originally discovered by the famous Indian mathematician Ramanujan [3]. We will derive them by a (possibly new) method found by the second author of this paper.

$$\sqrt[3]{\cos\alpha} + \sqrt[3]{\cos 2\alpha} + \sqrt[3]{\cos 3\alpha} = \sqrt[3]{\frac{5 - 3\sqrt[3]{7}}{2}}\,, \tag{7}$$

$$\text{and } \sqrt[3]{\sec\alpha} + \sqrt[3]{\sec 2\alpha} + \sqrt[3]{\sec 3\alpha} = \sqrt[3]{8 - 6\sqrt[3]{7}}\,. \tag{8}$$

It can be shown that the right hand sides of (7) and (8) are not constructible although they are relatively simple closed form expressions. (See [1].) They are however, clearly algebraic.

About 50 years ago, high school and college students learned a topic in elementary algebra called the "theory of equations". This topic is almost never taught today. Its subject was the roots of polynomial equations. Many of the techniques used in this paper would have been familiar to students of this theory of equations. The topics in section 2 of this paper should be accessible to good students of pre-calculus mathematics. In some cases, a number of elementary manipulations are omitted and the reader will have to work these out. The topics in section 3 will be very challenging even for good students.

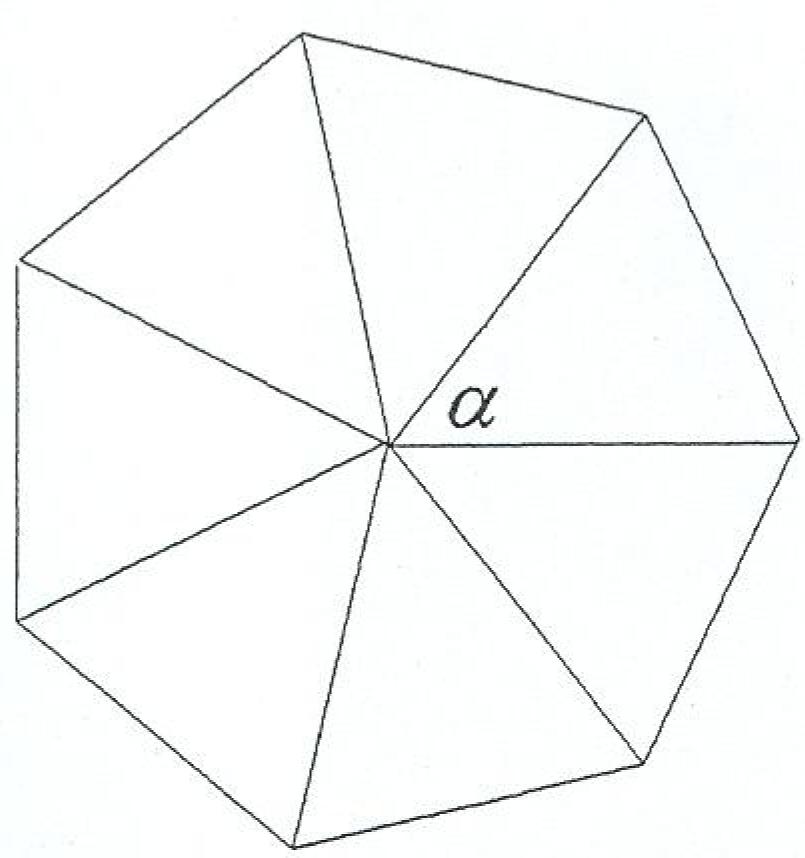

Figure 1: The regular heptagon

2. THE FUNDAMENTAL CUBIC

In this section we will prove (1) through (6). We begin with the following:

Theorem: *The roots of the cubic equation*

$$8x^3 + 4x^2 - 4x - 1 = 0 \tag{9}$$

are $x_1 = \cos\alpha$, $x_2 = \cos 2\alpha$, *and* $x_3 = \cos 3\alpha$, *where* $\alpha = 2\pi/7$.

Proof: Let $7\phi_n = 2\pi n$, where n is any integer. Then $3\phi_n = 2\pi n - 4\phi_n$ and

$$\cos 3\phi_n = \cos 4\phi_n. \tag{10}$$

Since $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ for all θ, (10) becomes $4\cos^3\phi_n - 3\cos\phi_n = 8\cos^4\phi_n - 8\cos^2\phi_n + 1$. This simplifies to $8x_n^4 - 4x_n^3 - 8x_n^2 + 3x_n + 1 = 0$, where we have set $x_n = \cos\phi_n$. For $n = 0,1,2,3$, we get $x_0 = 1$, $x_1 = \cos\alpha$, $x_2 = \cos 2\alpha$ and $x_3 = \cos 3\alpha$. Thus $x - 1$ divides $8x^4 - 4x^3 - 8x^2 + 3x + 1$ and we get

$$8x^4 - 4x^3 - 8x^2 + 3x + 1 = (x-1)\left(8x^3 + 4x^2 - 4x - 1\right).$$

This shows that x_1, x_2 and x_3 must be the roots of (9) and the theorem is proved.

We can write the roots of a cubic equation explicitly in terms of cube roots. This solution for the equation (9) is (1), (2) and (3). The reader interested in the details of this calculation can see [2] or any elementary text containing the theory of equations. Mathematical software such as *Mathematica* will also generate these roots.

From the elementary theory of equations we have the following:

Lemma 1: *If a, b and c are the roots of* $y^3 - Ay^2 + By - C = 0$, *then* $a + b + c = A$, $ab + ac + bc = B$ *and* $abc = C$.

Thus the roots of (9) give us the three relations

$$x_1 + x_2 + x_3 = -1/2, \tag{11}$$

$$x_2x_3 + x_1x_3 + x_1x_2 = -1/2, \tag{12}$$

$$\text{and } x_1x_2x_3 = 1/8. \tag{13}$$

From (11) and (13) we see the truth of (4) and (6) immediately. Using (13) to change (12) to $1/x_1 + 1/x_2 + 1/x_3 = -4$ we see that (5) is true.

We note that while $x_1 = \cos\alpha$, $x_2 = \cos 2\alpha$ and $x_3 = \cos 3\alpha$ are not constructible, the right hand sides of (11), (12) and (13) are clearly constructible. We think this is remarkable.

3. A SECOND CUBIC

To derive (7) and (8) we need a second cubic equation.

Theorem 2: *The cubic equation*

$$2y^3 - 2\sqrt[3]{\frac{5-3\sqrt[3]{7}}{2}}\,y^2 + \sqrt[3]{8-6\sqrt[3]{7}}\,y - 1 = 0 \qquad \textbf{(14)}$$

has roots $y_1 = \sqrt[3]{\cos\alpha}$, $y_2 = \sqrt[3]{\cos 2\alpha}$ *and* $y_3 = \sqrt[3]{\cos 3\alpha}$.

Proof: We will look for a cubic with roots $y_1 = \sqrt[3]{\cos\alpha}$, $y_2 = \sqrt[3]{\cos 2\alpha}$ and $y_3 = \sqrt[3]{\cos 3\alpha}$. From (6) we know that $y_1 y_2 y_3 = \sqrt[3]{x_1 x_2 x_3} = 1/2$. Thus, from Lemma 1, the desired cubic has the form

$$2y^3 - py^2 + qy - 1 = 0, \qquad \textbf{(15)}$$

where p and q are to be determined. We have at once $2y^3 - 1 = py^2 - qy$, so

$$\left(2y^3 - 1\right)^3 = \left(py^2 - qy\right)^3.$$

Expanding the right hand side we get

$$\left(2y^3 - 1\right)^3 = p^3y^6 - q^3y^3 - 3pqy^3\left(py^2 - qy\right).$$

But $py^2 - qy = 2y^3 - 1$ so this last equation becomes

$$\left(2y^3 - 1\right)^3 = p^3y^6 - q^3y^3 - 3pqy^3\left(2y^3 - 1\right).$$

If we let $y^3 = x$ and expand the left and right hand sides to get

$$8x^3 - 12x^2 + 6x - 1 = p^3x^2 - q^3x - 6pqx^2 + 3pqx.$$

Collecting terms with the same power of x we get the polynomial

$$8x^3 - \left(p^3 - 6pq + 12\right)x^2 + \left(q^3 - 3pq + 6\right)x - 1 = 0. \qquad \textbf{(16)}$$

By the definition of x, the equation (16) is the same as equation (9). Therefore, $p^3 - 6pq + 12 = -4$ and $q^3 - 3pq + 6 = -4$. These simplify to

$$p^3 - 6pq = -16, \qquad \textbf{(17)}$$

$$\text{and } q^3 - 3pq = -10. \qquad \textbf{(18)}$$

Our problem now is to solve the two equations (17) and (18) for the two quantities p and q. It is easy to eliminate pq and get

$$p^3 - 2q^3 = 4. \qquad \textbf{(19)}$$

(Considerable manipulation was required by the authors to continue the solution and obtain the final values for p and q. After obtaining the solution, and carefully examining its structure, we realized that a relatively simple substitution of the form

$$p^3 = 20 + 6t, \qquad \textbf{(20)}$$

where t is to be determined, considerably shortens the algebraic manipulation. Thus, we present (20), without motivation, as a magician takes a rabbit from a hat.)

From (19) and (20) we get

$$q^3 = 8 + 3t. \qquad \textbf{(21)}$$

From (17) and (20) we get $pq = 6 + t$, and so

$$p^3 q^3 = (6 + t)^3. \qquad \textbf{(22)}$$

Using (20) and (21) in (22) we get $(20 + 6t)(8 + 3t) = (6 + t)^3$, which simplifies to $t^3 = -56$ or $t = -2\sqrt[3]{7}$. Substituting this value of t in (20) and (21) we get at once

$$p = 2\sqrt[3]{\frac{5 - 3\sqrt[3]{7}}{2}}, \text{ and } q = \sqrt[3]{8 - 6\sqrt[3]{7}}.$$

Now (15) becomes (14) and the theorem is proved.

The truth of (7) and (8) follow from Theorem 2 using Lemma 1.

4. FINAL REMARKS

1. There appear to be more relations like (4) through (8). Using the method shown above in section 3, we invite the reader to verify that

$$\sqrt[3]{\cos^2 \alpha} + \sqrt[3]{\cos^2 2\alpha} + \sqrt[3]{\cos^2 3\alpha} = \frac{\sqrt[3]{22 + 12\sqrt[3]{7} + 6\sqrt[3]{49}}}{2},$$

$$\text{and } \sqrt[3]{\sec^2 \alpha} + \sqrt[3]{\sec^2 2\alpha} + \sqrt[3]{\sec^2 3\alpha} = \sqrt[3]{12\left(4 + 2\sqrt[3]{7} + \sqrt[3]{49}\right)}.$$

2. While we centered our attention on $\alpha = 2\pi/7$, the central angle of the regular heptagon, several of our results can be easily modified to the central angle $\alpha = 2\pi/(2n+1)$, of a regular $(2n+1)$-gon, where n = 4, 5, 6, … . The reader might want to try to generalize the methods used in section 2 to find more results like (4), (5) and (6).

REFERENCES

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