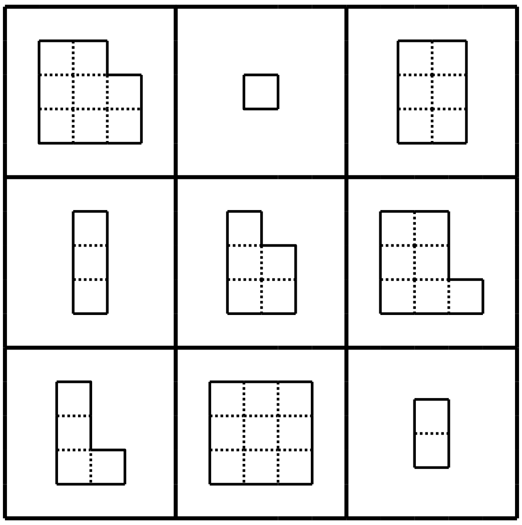
International Teenagers Mathematics Olympiad

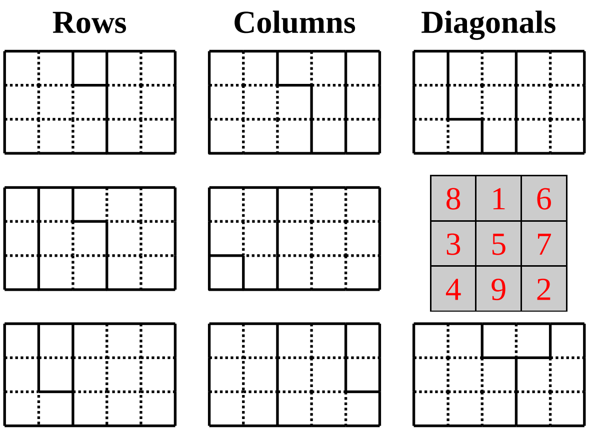
10 – 14 December, 2015, Sungai Petani Kedah Malaysia

TEAM CONTEST

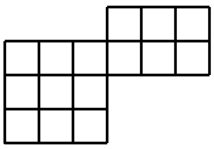
1. The geometric magic square in the diagram below on the left is based on the ordinary magic square in the diagram below on the right. The latter is shaded. Of course, the total number of unit squares in the three pieces in each row, column and diagonal is equal to 15, known as magic constant. If that is all, we are not doing anything new. Instead, the magic constant is no longer a number but a figure, which can be formed with the three pieces in each row, column and diagonal. Rotations and reflections of the pieces are allowed. The diagram below on the right shows that the magic constant can be a 3×5 rectangle.

Magic Square



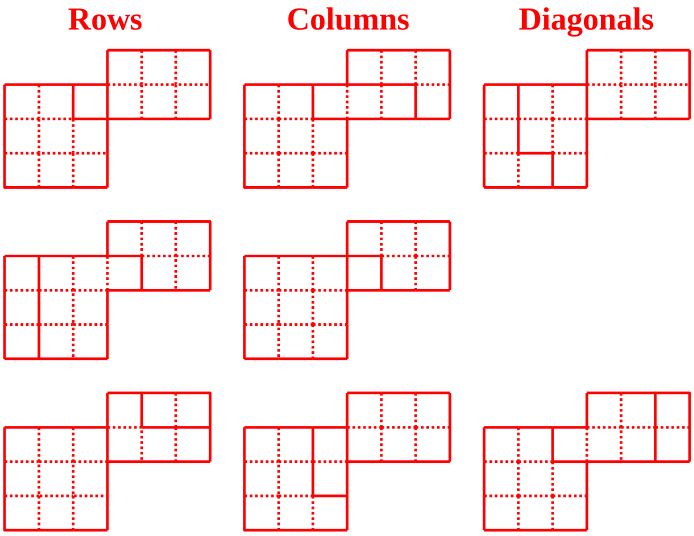


Show that the figure in the diagram below can also be the magic constant for the geometric magic square above. 【**Submitted by Jury**】



【***Solution***】

The constructions are shown in the right diagram.



2. Let a be a positive integer such that a^2+b^2-a is a multiple of ab for some positive integer b relatively prime to a. Find the maximum value of a.
【**Submitted by *SINGAPORE***】

【***Solution***】

Let $a^2+b^2-a=kab$. Since a and b are relatively prime , then b is a multiple of a, let $b=ah$. (10 points)

Hence $a^2+a^2h^2-a=ka^2h$.(10 points) It follows that $a+ah^2-1=kah$.(10 points)

It follows that a divides 1, and we must have a = 1. (10 points)

ANS: 1

3. Let $S(x)$ denotes the sum of the digits in the decimal representation of a positive integer x.
Find the largest value of x such that $x+S(x)+S(S(x))+S(S(S(x)))=2015$.
【**Submitted by *VIETNAM***】

【***Solution***】

Suppose that x is a positive integer satisfying the requirement of the problem. Obviously, x < 2015 and therefore $S(x)\le S(1999)=28$. This also shows that $S(S(x))\le S(19)=10$ and $S(S(S(x)))\le S(9)=9$. From this, we get

$$2015>x\ge 2015-(28+10+9)=1968.$$

On the other hand, we have

$$2015=x+S(x)+S(S(x))+S(S(S(x)))\equiv 4x\ (\text{mod } 9).$$

It follows that $4x\equiv 8\ (\text{mod } 9)$ or equivalently, $x\equiv 2\ (\text{mod } 9)$. Combining this with $2015>x\ge 1968$ from above, we obtain

$$x\in\{1973, 1982, 1991, 2000, 2009\}.$$

For x = 1973, $x+S(x)+S(S(x))+S(S(S(x)))=1973+20+2+2=1997$;

for x = 1982, $x+S(x)+S(S(x))+S(S(S(x)))=1982+20+2+2=2006$;

for x = 1991, $x+S(x)+S(S(x))+S(S(S(x)))=1991+20+2+2=2015$;

for x = 2000, $x+S(x)+S(S(x))+S(S(S(x)))=2000+2+2+2=2006$;

for x = 2009, $x+S(x)+S(S(x))+S(S(S(x)))=2009+11+2+2=2024$.

So, we conclude that x = 1991 is the only one solution of the problem.

ANS: 1991

4. A two-player game starts with the number 111 on the blackboard. Anna goes first, followed by Boris, taking turns alternately thereafter. In each move, Anna may reduce the number on the blackboard by 1 or 10, while Boris may reduce it by 1, 2, 8 or 10. The player who reduces the number to 0 wins. Give a winning strategy for Boris.【**Submitted by *CYPRUS***】

【***Solution***】

Boris wins if he always reduces the number by 2 or 8. (10 points) The starting number 111 is a multiple of 3. (10 points) After Anna's turn, the number will be 2 more than a multiple of 3. (10 points) If Boris sticks to his strategy, he will again leave behind a multiple of 3. Since 0 is a multiple of 3, Boris must win. (10 points)

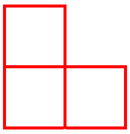
5. In how many different ways can you choose three squares in an 8×8 chessboard so that every two of them share at least one corner? 【*2013 AITMO Proposal*】

【***Solution* 1**】

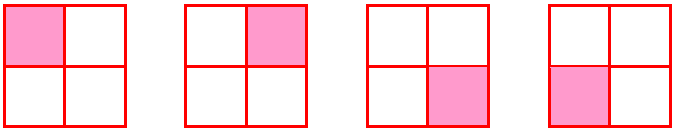
Clearly, of the three squares, the three squares are either two black and a white, or the other way round. We can focus on the first case. Of the 32 possible choices for the white square, 2 are at corners, 12 are on the edges and 18 are in the interior. In the first case, the black squares can be chosen in only 1 way. In the second case, they can be chosen in 2 ways. In the third case, they can be chosen in 4 ways. Hence the grand total is $2\times(2\times1+12\times2+18\times4)=196$.

【***Solution* 2**】

Any two of three Squares in the chessboard share at least one corner must be in the shape of V-Triomino.



In a 2×2 sub-chessboard, there are 4 possible choices, and there are $7\times7=49$ sub-board, hence we have $4\times49=196$ different ways.



ANS: 196

6. Each square of a 4×4 table is filled with a different one of the positive integers 1, 2, 3, …, 15, 16. For every two squares sharing a side, the numbers in them are added and the largest sum is recorded. What is the minimum value of this largest sum? 【**Submitted by *VIETNAM***】

【***Solution***】

If the number 16 is not at a corner, then it has at least 3 neighbours and the largest sum is at least $16+3=19$.(10 points) Suppose it is at a corner. If the number 15 is one of its neighbours, then the largest sum will be $16+15=31$.(10 points) Otherwise, 15 will have at least two neighbours which are not also neighbours of 16. It follows that the largest sum is at least $15+4=19$. (10 points)The table below shows that the largest sum can in fact be just 19.(Example 10 points)

16	1	11	8
2	15	4	10
14	3	12	7
5	13	6	9

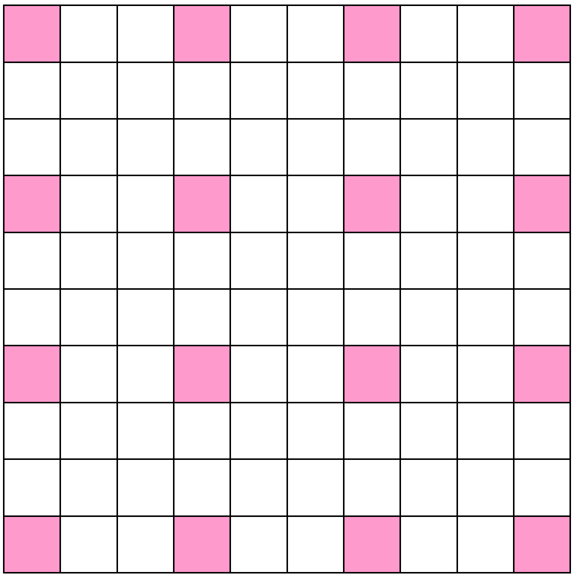
ANS: 19

7. A city is divided into 100 squares in a 10×10 configuration. Each police squad occupies two squares which share exactly one corner. What is the minimum number of police squads so that every square not occupied by a police squad must share a side with a square occupied by a police squad? 【**Submitted by *Russia***】

【***Solution***】

A square is said to be covered by a police squad if it is either occupied by that squad or shares a side with a square occupied by that squad. The diagram below on the left shows 16 shaded squares no two of which may be covered by the same police squad. It follows that 16 police squads are necessary. The diagram below on the right shows that 16 police squads are sufficient.

ANS: 16



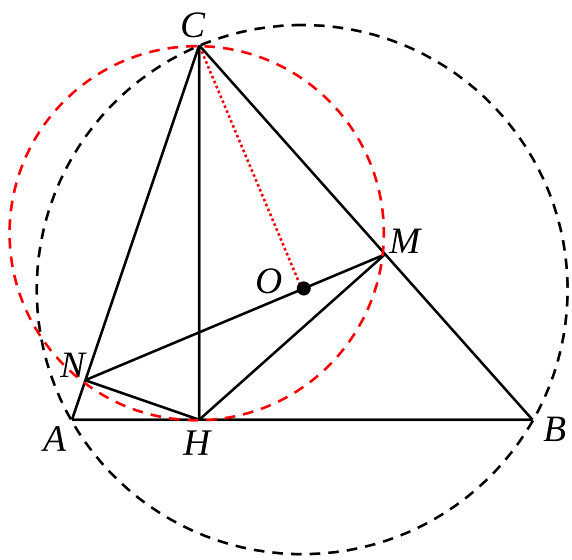
		13			12			10	
14			13	12			10		
	14					11			9
		15			11			9	
16			15					8	
	16					7			8
	1			3			7		
1			3					6	
		2			4	5			6
	2			4			5		

8. ABC is an acute triangle. H is the foot of the altitude from C to AB. M and N are the respective feet of perpendicular from H to BC and CA. The circumcentre of ABC lies on MN. If the circumradius of ABC is $\sqrt{2}$ cm, what is the length of CH, in cm? 【**Submitted by *Bulgaria_SMG***】

【***Solution***】

CH is a diameter of the circumcircle of triangle CMN. Hence $\angle CHN=\angle CMN$.(10 points) We also have $\angle CHN=90^\circ-\angle NHA=\angle CAB$. Hence $\angle CMN=\angle CAB$.(10 points) Similarly, $\angle CNM=\angle CBA$, so that triangles CMN and CAB are similar. (10 points) Since $CO=\sqrt{2}$ cm is the circumradius of triangle ABC, we have

$$\frac{\sqrt{2}}{CH}=\frac{\frac{1}{2}CH}{\sqrt{2}}$$ so that $CH=2$ cm. (10 points)



ANS: 2

9. Find the number of ways of colouring 12 different squares of a 6×4 chessboard such that there are two coloured squares in each row and three in each column. 【**Submitted by *CYPRUS***】

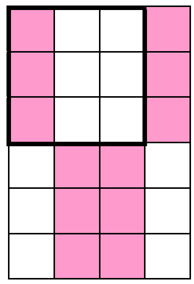
【***Solution***】

Consider the first column. There are $C_3^6=20$ ways to choose the 3 rows in which the square will be coloured and place them first rows.

Now consider the 3×3 table which contains the 3 coloured squares , one square in each row. The fourth column is determined with all squares coloured

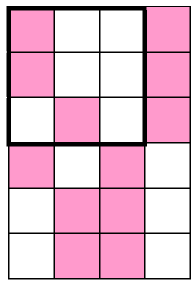
We study three cases.

1st Case:



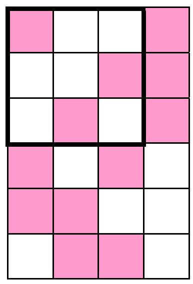
The three coloured squares are in the same column. In this case there are 3 choices for which column that is. The bottom half of board is fixed. There is no choices.

2nd Case:



Two coloured squares are in the same column and one is in the other. In this case, there are 3 ways to choose the column to colour the 2 squares, and 2 ways to choose which one gets the other coloured square. Then there are 3 ways to choose which row the one coloured square is in. Now, in the bottom half of board, the 3 squares in the column with no coloured squares in the top half must all be coloured, so there are no choices. In the column with two coloured squares in the top half, we can choose any of the 3 squares to colour it. For the last two squares we have not any choices. So we have $3\times 2\times 3\times 3=54$.

3rd Case:



The three coloured squares are in a different columns. So, for the first column there are 3 ways to choose the square to be coloured and 2 ways to choose the square in the second column. For the third column we have no choice. Again we think what happen in the bottom half of the board. Must 2 squares be coloured in each row and column, 3 ways for the first column, 2 ways for the second , and none for the third. So we have $3\times 2\times 3\times 2=36$.

So, Costas has $2\times(3+54+36)=1860$ different ways to colour the 12 squares of the board.

ANS: 1860

10. What is the minimum number of fourth powers of integers, not necessarily distinct, such that their sum is 2015? 【**Submitted by *BURGAS***】

【***Solution***】

Since $7^4 = 2401 > 2015$, we can only use the fourth powers of 1, 2, 3, 4, 5 and 6. (10 points) The Greedy Algorithm yields $6^4 + 5^4 + 3^4$ plus 13 copies of 1 for a sum of 16 fourth powers. (10 points) Suppose we use less than 16 fourth powers. The fourth power of an even number is a multiple of 16 while the fourth power of an odd number is 1 more than a multiple of 16. Since 2015 is 15 more than a multiple of 16, we must use exactly 15 fourth powers of odd integers. (10 points) If we use at most 1 copy of $5^4 = 625$, adding 14 copies of $3^4 = 81$ will not be enough. If we use 2 copies of 625, we can use at most 9 copies of 81 and must use at least 36 copies of 1. If we use 3 copies of 625, we can use at most 1 copy of 81 and must use at least 59 copies of 1. We cannot use 4 or more copies of 625. It follows that we must use at least 16 fourth powers. (10 points)

ANS: 16